Exercise 2

A particle moves according to a law of motion $s = f(t)$, $t \geq 0$, where t is measured in seconds and s in feet.

(a) Find the velocity at time t.

(b) What is the velocity after 1 second?

(c) When is the particle at rest?

(d) When is the particle moving in the positive direction?

(e) Find the total distance traveled during the first 6 seconds.

(f) Draw a diagram like Figure 2 to illustrate the motion of the particle.

(g) Find the acceleration at time t and after 1 second.

(h) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 6$.

(i) When is the particle speeding up? When is it slowing down?

$$f(t) = \frac{9t}{t^2 + 9}$$

Solution

Part (a)

To find the velocity, take the derivative of the position function.

$$\begin{aligned} v(t) &= \frac{ds}{dt} \\ &= \frac{d}{dt}\left(\frac{9t}{t^2+9}\right) \\ &= \frac{\left[\frac{d}{dt}(9t)\right](t^2+9) - (9t)\left[\frac{d}{dt}(t^2+9)\right]}{(t^2+9)^2} \\ &= \frac{(9)(t^2+9) - (9t)(2t)}{(t^2+9)^2} \\ &= \frac{81 - 9t^2}{(t^2+9)^2} \end{aligned}$$

Part (b)

The velocity after 1 second has elapsed is

$$v(1) = \frac{81 - 9(1)^2}{(1^2+9)^2} = \frac{72}{100} = \frac{18}{25}\ \frac{\text{feet}}{\text{second}}.$$

Part (c)

To find when the particle is at rest, set the velocity function equal to zero and solve the equation for t.

$$v(t) = 0$$

$$\frac{81 - 9t^2}{(t^2 + 9)^2} = 0$$

$$81 - 9t^2 = 0$$

$$9(9 - t^2) = 0$$

$$t = \{-3, 3\}$$

Since $t \geq 0$, the particle is at rest when $t = 3$.

Part (d)

To find when the particle is moving in the positive direction, find what values of t satisfy $v(t) > 0$.

$$v(t) > 0$$

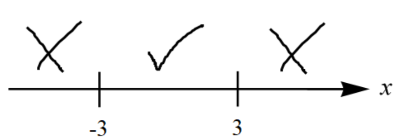
$$\frac{81 - 9t^2}{(t^2 + 9)^2} > 0$$

$$81 - 9t^2 > 0$$

$$9(9 - t^2) > 0$$

$$9(3 + t)(3 - t) > 0$$

The critical values are -3 and 3. Partition the number line at these points and test whether the inequality is true within each of the intervals.



Therefore, the particle is moving in the positive direction for $0 \leq t < 3$.

Part (e)

The distance travelled in $0 \leq t < 3$ is

$$|s(3) - s(0)| = \left| \frac{9(3)}{(3)^2 + 9} - \frac{9(0)}{(0)^2 + 9} \right| = \frac{3}{2},$$

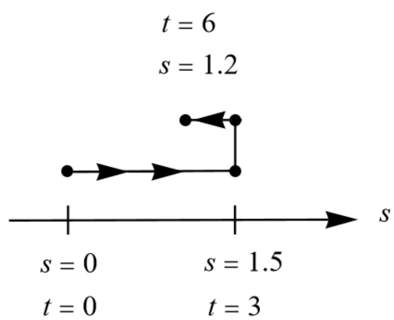
and the distance travelled in $3 < t \leq 6$ is

$$|s(6) - s(3)| = \left|\frac{9(6)}{(6)^2 + 9} - \frac{9(3)}{(3)^2 + 9}\right| = \frac{3}{10}.$$

Consequently, the total distance travelled in $0 \leq t \leq 6$ is $\frac{3}{2} + \frac{3}{10} = \frac{9}{5}$ feet.

Part (f)

Below is an illustration of the particle's motion from $t = 0$ to $t = 6$.



Part (g)

Calculate the derivative of the velocity to get the acceleration.

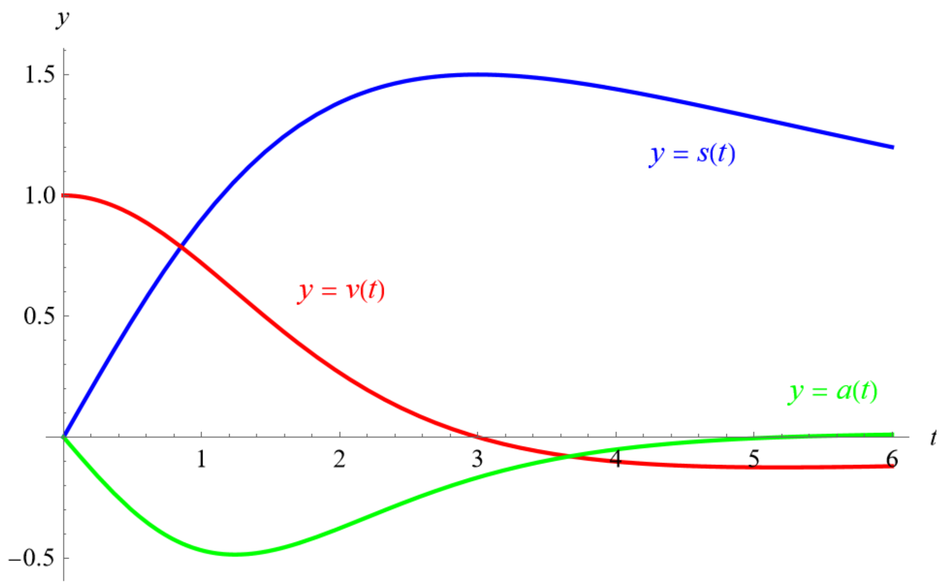
$$\begin{aligned} a(t) &= \frac{dv}{dt} \\ &= \frac{d}{dt}\left[\frac{81 - 9t^2}{(t^2 + 9)^2}\right] \\ &= \frac{\left[\frac{d}{dt}(81 - 9t^2)\right](t^2 + 9)^2 - (81 - 9t^2)\left\{\frac{d}{dt}[(t^2 + 9)^2]\right\}}{(t^2 + 9)^4} \\ &= \frac{(-18t)(t^2 + 9)^2 - (81 - 9t^2)[2(t^2 + 9) \cdot 2t]}{(t^2 + 9)^4} \\ &= \frac{18t^5 - 324t^3 - 4374t}{(t^2 + 9)^4} \end{aligned}$$

The acceleration after 1 second is

$$a(1) = \frac{18(1)^5 - 324(1)^3 - 4374(1)}{(1^2 + 9)^4} = -\frac{117}{250}\ \frac{\text{feet}}{\text{second}^2}.$$

Part (h)

Below is a plot of the position, velocity, and acceleration versus time for $0 \leq t \leq 6$.



Part (i)

The particle is speeding up when

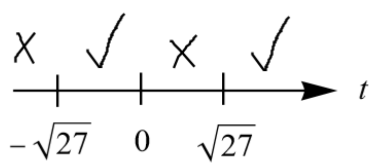
$$\frac{18t^5 - 324t^3 - 4374t}{(t^2+9)^4} > 0$$

$$18t^5 - 324t^3 - 4374t > 0$$

$$18t(t^2 - 27)(t^2+9) > 0$$

$$18t(t+\sqrt{27})(t-\sqrt{27})(t^2+9) > 0.$$

The critical values are $-\sqrt{27}$, 0, and $\sqrt{27}$. Partition the number line at these points and test whether this inequality is true within each interval.



Consequently, the particle is speeding up when $\sqrt{27} < t \leq 6$.

The particle is slowing down when

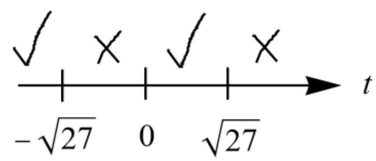
$$\frac{18t^5 - 324t^3 - 4374t}{(t^2+9)^4} < 0$$

$$18t^5 - 324t^3 - 4374t < 0$$

$$18t(t^2 - 27)(t^2 + 9) < 0$$

$$18t(t + \sqrt{27})(t - \sqrt{27})(t^2 + 9) < 0.$$

The critical values are $-\sqrt{27}$, 0, and $\sqrt{27}$. Partition the number line at these points and test whether this inequality is true within each interval.



Consequently, the particle is slowing down when $0 \leq t < \sqrt{27}$.